SL Paper 2

The set of all integer s from 0 to 99 inclusive is denoted by *S*. The binary operations $*$ and $\circ$ are defined on *S* by

$$a * b = [a + b + 20] \text{(mod 100)}$$
$$a \circ b = [a + b - 20] \text{(mod 100)}.$$

The equivalence relation *R* is defined by $aRb \Leftrightarrow \left(\sin\frac{\pi a}{5} = \sin\frac{\pi b}{5}\right)$.

a. Find the identity element of *S* with respect to $*$. [3]

b. Show that every element of *S* has an inverse with respect to $*$. [2]

c. State which elements of *S* are self-inverse with respect to $*$. [2]

d. Prove that the operation $\circ$ is not distributive over $*$. [5]

e. Determine the equivalence classes into which *R* partitions *S*, giving the first four elements of each class. [5]

f. Find two elements in the same equivalence class which are inverses of each other with respect to $*$. [2]

Consider the set $J = \left\{a + b\sqrt{2} : a,\ b \in \mathbb{Z}\right\}$ under the binary operation multiplication.

Consider $a + b\sqrt{2} \in G$, where $\gcd(a,\ b) = 1$,

a. Show that J is closed. [2]

b. State the identity in J. [1]

c. Show that [5]

(i) $1 - \sqrt{2}$ has an inverse in J;

(ii) $2 + 4\sqrt{2}$ has no inverse in J.

d. Show that the subset, G, of elements of J which have inverses, forms a group of infinite order. [7]

e. (i) Find the inverse of $a + b\sqrt{2}$. [4]

(ii) Hence show that $a^2 - 2b^2$ divides exactly into a and b.

(iii) Deduce that $a^2 - 2b^2 = \pm 1$.

a. (i) Draw the Cayley table for the set $S = \{0, 1, 2, 3, 4, 5\}$ under addition modulo six $(+_6)$ and hence show that $\{S, +_6\}$ is a group. [11]

(ii) Show that the group is cyclic and write down its generators.

(iii) Find the subgroup of $\{S, +_6\}$ that contains exactly three elements.

b. Prove that a cyclic group with exactly one generator cannot have more than two elements. [4]

c. H is a group and the function $\Phi : H \to H$ is defined by $\Phi(a) = a^{-1}$, where a^{-1} is the inverse of $\boldsymbol{a}$ under the group operation. Show that Φ is an isomorphism **if and only if *H*** is Abelian. [9]

The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is defined by $\boldsymbol{X} \mapsto \boldsymbol{AX}$, where $\boldsymbol{X} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a , b , c , d are all non-zero.

Consider the group $\{S, +_m\}$ where $S = \{0, 1, 2 \ldots m - 1\}$, $m \in \mathbb{N}$, $m \geq 3$ and $+_m$ denotes addition modulo m .

A.a. Show that f is a bijection if $\boldsymbol{A}$ is non-singular. [7]

A.b. Suppose now that $\boldsymbol{A}$ is singular. [5]

(i) Write down the relationship between a , b , c , d .

(ii) Deduce that the second row of $\boldsymbol{A}$ is a multiple of the first row of $\boldsymbol{A}$.

(iii) Hence show that f is not a bijection.

B.a. Show that $\{S, +_m\}$ is cyclic for all $\boldsymbol{m}$. [3]

B.b. Given that m is prime, [7]

(i) explain why all elements except the identity are generators of $\{S, +_m\}$;

(ii) find the inverse of x , where $\boldsymbol{x}$ is any element of $\{S, +_m\}$ apart from the identity;

(iii) determine the number of sets of two distinct elements where each element is the inverse of the other.

B.c. Suppose now that $m = ab$ where a , b are unequal prime numbers. Show that $\{S, +_m\}$ has two proper subgroups and identify them. [3]

The binary operator $*$ is defined for $\boldsymbol{a}$, $b \in \mathbb{R}$ by $a * b = a + b - ab$.

a. (i) Show that $*$ is associative. [15]

(ii) Find the identity element.

(iii) Find the inverse of $a \in \mathbb{R}$, showing that the inverse exists for all values of a except one value which should be identified.

(iv) Solve the equation $x * x = 1$.

b. The domain of $*$ is now reduced to $S = \{0, 2, 3, 4, 5, 6\}$ and the arithmetic is carried out modulo 7. [17]

(i) Copy and complete the following Cayley table for $\{S, *\}$.

$*$	**0**	**2**	**3**	**4**	**5**	**6**
0	0	2	3	4	5	6
2	2	0	6	5	4	3
3	3					
4	4					
5	5					
6	6					

(ii) Show that $\{S, *\}$ is a group.

(iii) Determine the order of each element in ***S*** and state, with a reason, whether or not $\{S, *\}$ is cyclic.

(iv) Determine all the proper subgroups of $\{S, *\}$ and explain how your results illustrate Lagrange's theorem.

(v) Solve the equation $2 * x * x = 5$.

The set S consists of real numbers ***r*** of the form $r = a + b\sqrt{2}$, where $a, b \in \mathbb{Z}$.

The relation R is defined on S by $r_1 R r_2$ if and only if $a_1 \equiv a_2$ (mod2) and $b_1 \equiv b_2$ (mod3), where $r_1 = a_1 + b_1\sqrt{2}$ and $r_2 = a_2 + b_2\sqrt{2}$.

a. Show that R is an equivalence relation. [7]

b. Show, by giving a counter-example, that the statement $r_1 R r_2 \Rightarrow r_1^2 R r_2^2$ is false. [3]

c. Determine [3]

(i) the equivalence class E containing $1 + \sqrt{2}$;

(ii) the equivalence class F containing $1 - \sqrt{2}$.

d. Show that [4]

(i) $(1 + \sqrt{2})^3 \in F$;

(ii) $(1 + \sqrt{2})^6 \in E$.

e. Determine whether the set E forms a group under [4]

(i) the operation of addition;

(ii) the operation of multiplication.

The set $S_n = \{1,\ 2,\ 3,\ \ldots,\ n-2,\ n-1\}$, where n is a prime number greater than 2, and $\times_n$ denotes multiplication modulo n.

a.i. Show that there are no elements $a,\ b \in S_n$ such that $a \times_n b = 0$. [2]

a.ii. Show that, for $a,\ b,\ c \in S_n,\ a \times_n b = a \times_n c \Rightarrow b = c$. [2]

b. Show that $G_n = \{S_n,\ \times_n\}$ is a group. You may assume that $\times_n$ is associative. [4]

c.i. Show that the order of the element $(n-1)$ is 2. [1]

c.ii. Show that the inverse of the element 2 is $\frac{1}{2}(n+1)$. [2]

c.iii. Explain why the inverse of the element 3 is $\frac{1}{3}(n+1)$ for some values of n but not for other values of n. [2]

c.iv. Determine the inverse of the element 3 in G_{11}. [1]

c.v. Determine the inverse of the element 3 in G_{31}. [2]

The set of all permutations of the list of the integers $1,\ 2,\ 3\ \ldots\ n$ is a group, S_n, under the operation of composition of permutations.

Each element of S_4 can be represented by a 4×4 matrix. For example, the cycle $(1\ 2\ 3\ 4)$ is represented by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \text{ acting on the column vector } \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

a. (i) Show that the order of S_n is $n!$; [9]

(ii) List the 6 elements of S_3 in cycle form;

(iii) Show that S_3 is not Abelian;

(iv) Deduce that S_n is not Abelian for $n \geqslant 3$.

b. (i) Write down the matrices $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ representing the permutations $(1\ 2),\ (2\ 3)$, respectively; [7]

(ii) Find $\boldsymbol{M}_1\boldsymbol{M}_2$ and state the permutation represented by this matrix;

(iii) Find $\det(\boldsymbol{M}_1)$, $\det(\boldsymbol{M}_2)$ and deduce the value of $\det(\boldsymbol{M}_1\boldsymbol{M}_2)$.

c. (i) Use mathematical induction to prove that [8]

$(1\ n)(1\ n\ -1)(1\ n-2)\ldots(1\ 2) = (1\ 2\ 3\ldots n)\ n \in \mathbb{Z}^+,\ n > 1.$

(ii) Deduce that every permutation can be written as a product of cycles of length 2.

Let f be a homomorphism of a group G onto a group H.

a. Show that if e is the identity in G, then $f(e)$ is the identity in H. [2]

b. Show that if x is an element of G, then $f(x^{-1}) = (f(x))^{-1}$. [2]

c. Show that if G is Abelian, then H must also be Abelian. [5]

d. Show that if S is a subgroup of G, then $f(S)$ is a subgroup of H. [4]

Consider the special case in which $G = \{1,\ 3,\ 4,\ 9,\ 10,\ 12\}$, $H = \{1,\ 12\}$ and $*$ denotes multiplication modulo 13.

a. The group $\{G,\ *\}$ has a subgroup $\{H,\ *\}$. The relation R is defined such that for $x, y \in G, xRy$ if and only if $x^{-1} * y \in H$. Show that R is an equivalence relation. [8]

b.i. Show that $3R10$. [4]

b.ii. Determine the three equivalence classes. [3]

S is defined as the set of all 2×2 non-singular matrices. A and B are two elements of the set S.

a. (i) Show that $(A^T)^{-1} = (A^{-1})^T$. [8]

(ii) Show that $(AB)^T = B^T A^T$.

b. A relation R is defined on S such that A is related to B if and only if there exists an element X of S such that $XAX^T = B$. Show that R is an equivalence relation. [8]

A group has exactly three elements, the identity element e, h and k. Given the operation is denoted by $\otimes$, show that

A.a. (i) Show that $\mathbb{Z}_4$ (the set of integers modulo 4) together with the operation $+_4$ (addition modulo 4) form a group G. You may assume associativity. [9]

(ii) Show that G is cyclic.

A.b. Using Cayley tables or otherwise, show that G and $H = (\{1, 2, 3, 4\}, \times_5)$ are isomorphic where $\times_5$ is multiplication modulo 5. State clearly all the possible bijections. [7]

B.b. the group is cyclic. [3]

b. the group is cyclic. [5]

A.a. The relation R_1 is defined for $a, b \in \mathbb{Z}^+$ by aR_1b if and only if $n \,|\, (a^2 - b^2)$ where n is a fixed positive integer. [11]

(i) Show that R_1 is an equivalence relation.

(ii) Determine the equivalence classes when $n = 8$.

B. Consider the group $\{G, *\}$ and let H be a subset of G defined by [12]

$$H = \{x \in G \text{ such that } x * a = a * x \text{ for all } a \in G\}.$$

Show that $\{H, *\}$ is a subgroup of $\{G, *\}$.

B.b. The relation R_2 is defined for $a, b \in \mathbb{Z}^+$ by aR_2b if and only if $(4 + |a - b|)$ is the square of a positive integer. Show that R_2 is not transitive. [3]

The relation R is defined on $\mathbb{R}^+ \times \mathbb{R}^+$ such that $(x_1, y_1)R(x_2, y_2)$ if and only if $\frac{x_1}{x_2} = \frac{y_2}{y_1}$.

a. Show that R is an equivalence relation. [6]

b. Determine the equivalence class containing (x_1, y_1) and interpret it geometrically. [3]

The set S contains the eighth roots of unity given by $\left\{\text{cis}\left(\frac{n\pi}{4}\right),\ n \in \mathbb{N},\ 0 \leqslant n \leqslant 7\right\}$.

(i) Show that $\{S, \times\}$ is a group where $\times$ denotes multiplication of complex numbers.

(ii) Giving a reason, state whether or not $\{S, \times\}$ is cyclic.

The binary operation multiplication modulo 9, denoted by $\times_9$, is defined on the set $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

a. Copy and complete the following Cayley table. [3]

$\times_9$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	4	6	8	1	3	5	7
3								
4	4	8	3	7	2	6	1	5
5								
6	6	3	0	6	3	0	6	3
7								
8	8	7	6	5	4	3	2	1

b. Show that $\{S, \times_9\}$ is not a group. [1]

c. Prove that a group $\{G, \times_9\}$ can be formed by removing two elements from the set S. [5]

d. (i) Find the order of all the elements of G. [8]

(ii) Write down all the proper subgroups of $\{G, \times_9\}$.

(iii) Determine the coset containing the element 5 for each of the subgroups in part (ii).

e. Solve the equation $4 \times_9 x \times_9 x = 1$. [3]

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a. The relation R is defined for $x, y \in \mathbb{Z}^+$ such that xRy if and only if $3^x \equiv 3^y (\bmod 10)$. [11]

(i) Show that R is an equivalence relation.

(ii) Identify all the equivalence classes.

b. Let S denote the set $\left\{x \,\middle|\, x = a + b\sqrt{3}, a, b \in \mathbb{Q}, a^2 + b^2 \neq 0\right\}$. [15]

(i) Prove that S is a group under multiplication.

(ii) Give a reason why S would not be a group if the conditions on a, b were changed to $a, b \in \mathbb{R}, a^2 + b^2 \neq 0$.
